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# A PROPOSITIONAL LOGIC WITH SUBJUNCTIVE CONDITIONALS

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In this paper a formalized logic of propositions, $P_{A1}$, is presented. It is proven consistent and its relationships to traditional logic, to PM ([15]), to subjunctive (including contrary-to-fact) implication and to the "paradoxes" of material and strict implication are developed. Apart from any intrinsic merit it possesses, its chief significance lies in demonstrating the feasibility of a general logic containing the *principle of subjunctive contrariety*, i.e., the principle that 'If $p$ were true then $q$ would be true' and 'If $p$ were true then $q$ would be false' are incompatible.

**1. Introduction.** The logistic system A1 — in *Section* 2 below — was developed under the following objective: to construct a formalized logic of propositions, $P_A$, which would satisfy the following requirements:

(1) $P_A$ would contain all the theorems of PM's propositional calculus, *with the proviso that*

(2) The sign '$\supset$' would *not* be interpreted as "if ... then ..." in $P_A$, but only as an abbreviation of expressions containing 'and' and 'not,' or of expressions containing 'or' and 'not,' or of expressions containing stroke functions.

(3) The expression "if ... then ..." would be assigned as the interpretation of a primitive symbol '$\rightarrow$'.

(4) As many as possible of the traditional principles of propositional logic — e.g., affirming the antecedent, denying the consequent, dilemmas, etc., as well as principles of immediate inference like double negation, transposition, etc. — would be expressed by theorems of $P_A$ with '$\rightarrow$' representing appropriate conditional components.

(5) The so-called "paradoxes of material implication" and "paradoxes of strict implication" would be provably excluded and without analogues among the hypothetical theorems of $P_A$.

(6) $P_A$ would include as theorems a class of principles involving "if ... then ..." which, though not theorems in PM and equivalent systems, could plausibly lay claim to logical truth for subjunctive conditionals — e.g., "It is false that if $p$ were true then $p$ would be false" and "It is false that both if $p$ were true $q$ would be true and if $p$ were true $q$ would be false."

(7) $P_A$ would be provably consistent in the sense that not both S and $-$S would be derivable as theorems.

The purpose of this paper is to investigate the extent to which the formalized calculus of propositions, $P_{A1}$, meets the requirements laid down

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for $P_A$. *Section 2* below outlines the formal calculus, A1, gives a proof of its consistency, and provides semantic rules which make it a formalized logic of propositions, $P_{A1}$; these pertain to requirements (2), (3), and (7). *Section 3* deals with the fourth requirement, showing the derivability in $P_{A1}$ of principles of traditional propositional logic. *Section 4* proves that the theorems of PM's propositional calculus are contained in $P_{A1}$ (though with a restriction on their interpretation), thus satisfying requirement (1). *Section 5* shows the derivation of a class of new principles (requirement (6)) compatible with subjunctive conditionality but sometimes incompatible with the "if ... then ..." of most preceding systems. *Section 6* establishes the absence of "paradoxes" of strict or material implication and the absence of analogues of these "paradoxes" for '$\rightarrow$', thus meeting requirement (5). In *Section 7* we discuss certain residual problems relating to whether $P_{A1}$ includes either too much, or too little, in its class of derivable formulas.

**2. The consistency of $P_{A1}$ and its interpretation.** The logistic system, A1, has the following elements and rules:

*I. Primitive Symbols*

1) Grouping devices: ( )
2) Constants: $-$, $.$, $\rightarrow$
3) Variables: $p, q, r, s, p_1, q_1, r_1, \ldots$

*II. Rules of Formation*

$F_1$. A single variable by itself is a wff.
$F_2$. If any formula S is a wff, then $-$S is a wff.
$F_3$. If S and S′ are wffs, then (S.S′) is a wff.
$F_4$. If S and S′ are wffs, then (S $\rightarrow$ S′) is a wff.

*III. Abbreviations*

$D_1$. $(S \vee S') = \text{df} -(-S.-S')$
$D_2$. $(S \supset S') = \text{df} -(S.-S')$
$D_3$. $(S \equiv S') = \text{df} ((S \supset S').(S' \supset S))$

*IV. Primitive Formulas*

$A_1$. $((q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
$A_2$. $((p \rightarrow q) \rightarrow ((r.p) \rightarrow (q.r)))$
$A_3$. $((p \rightarrow -(q.r)) \rightarrow ((q.p) \rightarrow -r))$
$A_4$. $((p.(q.r)) \rightarrow (q.(p.r)))$
$A_5$. $((p \rightarrow -q) \rightarrow (q \rightarrow -p))$
$A_6$. $(--p \rightarrow p)$
$A_7$. $((p \rightarrow q) \rightarrow -(p.-q))$
$A_8$. $-((p.q).-p)$
$A_9$. $-(p.-(p.p))$
$A_{10}$. $((p \rightarrow q) \rightarrow -(p \rightarrow -q))$

*V. Rules of Transformation*

$R_1$. If ⊢ S and ⊢ (S → S′), then ⊢ S′.

$R_2$. If ⊢ S and ⊢ S′, then ⊢ (S.S′).

$R_3$. If ⊢ S and if *v* is a propositional variable occurring in S, then if S′ is got by replacing all occurrences of *v* in S by any wff, T, then ⊢ S′.

$R_4$. If ⊢ S and S′ is got by replacing any part, or all, of S by an expression equivalent through rules of abbreviation, then ⊢ S′.

The consistency of this system is established by the following matrices:

| $-$ | $p$ |
|---|---|
| 3 | 0 |
| 2 | 1 |
| 1 | 2 |
| 0 | 3 |

| $(p.q)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 0 | 3 | 2 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 3 | 2 | 3 | 2 |
| 3 | 2 | 3 | 2 | 3 |

| $(p \rightarrow q)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 2 |
| 1 | 2 | 1 | 2 | 3 |
| 2 | 1 | 2 | 1 | 2 |
| 3 | 2 | 1 | 2 | 1 |

The values 0 and 1 are designated values. Each primitive formula takes a designated value under every possible assignment of values to its variables. Rules of transformation preserve this property in all derived formulas; this is established for $R_1$ and $R_2$ by consideration of the matrices above, and for $R_3$ and $R_4$ by virtue of the same considerations which show that these rules of transformation preserve tautology in standard propositional calculi.

The purely formal system, A1, becomes a formalized logic of propositions $P_{A1}$, when it is supplemented by the following semantic rules which give an interpretation to its primitive symbols and so by implication to all its symbols: 1) the variables *p*, *q*, etc., shall be taken as propositional variables having propositions as values, 2) the constant, '—' shall be interpreted as "it is false that ...", 3) the constant '.' shall be interpreted as "and," 4) the constant '→' shall be interpreted as "if ... then ....".

With these interpretations, we now speak of Abbreviations of A1, as Definitions in $P_{A1}$; Primitive Formulas of A1 as the Axioms, $A_1$ to $A_{10}$, of $P_{A1}$; and Rules of Transformation of A1 as the Rules of Inference in the propositional logic, $P_{A1}$.

**3. Derivation of traditional principles of logic.** In this section we show the derivability of a sufficient number of traditional principles of logic to satisfy the demand that traditional logic be incorporated in the axiomatization. Since we can not move from $-(p.-q)$ or $(-p \vee q)$ to $(p \rightarrow q)$, the derivation of principles involving "if ... then ..." in this logic must often follow different paths from those available in PM. For this reason, proofs are given in detail.

First we derive the converse of Axiom 6, thus establishing both principles of *Double Negation*:

| | | |
|---|---|---|
| *1 | $(---p \rightarrow -p)$ | $[A_6\ p/-p]$ |
| *2 | $(p \rightarrow ----p)$ | $[{}^*1, A_5\ p/---p, q/p; R_1]$ |

*3 $(----p \rightarrow --p)$ [$A_6$ $p/--p$]

*4 $((p \rightarrow ----p) \rightarrow (p \rightarrow --p))$ [*3, $A_1$ $q/----p, r/--p$; $R_1$]

*5 $(p \rightarrow --p)$ [*2, *4, $R_1$].

Next, we establish $\vdash (p \rightarrow p)$:

*6 $((p \rightarrow --p) \rightarrow (p \rightarrow p))$ [$A_6$, $A_1$ $q/--p, r/p$; $R_1$]

*7 $(p \rightarrow p)$ [*5, *6; $R_1$].

The next eleven proofs establish, among others, the three principles *13, *15, and *19 which, with $A_5$, constitute the *Laws of Transposition*:

*8 $((q \rightarrow --p) \rightarrow (q \rightarrow p))$ [$A_6$, $A_1$ $q/--p, r/p, p/q$; $R_1$]

*9 $((q \rightarrow p) \rightarrow (q \rightarrow --p))$ [*5, $A_1$ $q/p, r/--p, p/q$; $R_1$]

*10 $((-p \rightarrow -q) \rightarrow (q \rightarrow --p))$ [$A_5$, $p/-p$]

*11 $((q \rightarrow --p) \rightarrow (-p \rightarrow -q))$ [$A_5$, $p/q, q/-p$]

*12 $(((-p \rightarrow -q) \rightarrow (q \rightarrow --p)) \rightarrow ((-p \rightarrow -q) \rightarrow (q \rightarrow p)))$
[*8, $A_1$ $q/(q \rightarrow --p), r/(q \rightarrow p), p/(-p \rightarrow -q)$; $R_1$]

*13 $((-p \rightarrow -q) \rightarrow (q \rightarrow p))$ [*10, *12; $R_1$]

*14 $(((q \rightarrow p) \rightarrow (q \rightarrow --p)) \rightarrow ((q \rightarrow p) \rightarrow (-p \rightarrow -q)))$
[*11, $A_1$ $q/(q \rightarrow --p), r/(-p \rightarrow -q), p/(q \rightarrow p)$; $R_1$]

*15 $((q \rightarrow p) \rightarrow (-p \rightarrow -q))$ [*9, *14; $R_1$]

*16 $((-q \rightarrow p) \rightarrow (-p \rightarrow --q))$ [*15 $q/-q$]

*17 $((-p \rightarrow --q) \rightarrow (-p \rightarrow q))$ [*8 $q/-p, p/q$]

*18 $(((-q \rightarrow p) \rightarrow (-p \rightarrow --q)) \rightarrow ((-q \rightarrow p) \rightarrow (-p \rightarrow q)))$
[*17, $A_1$ $q/(-p \rightarrow --q), r/(-p \rightarrow q), p/(-q \rightarrow p$; $R_1$]

*19 $((-q \rightarrow p) \rightarrow (-p \rightarrow q))$ [*16, *18; $R_1$]

Principles of *Commutation*, or *Permutation*, for conjunction and disjunction follow:

*20 $((q.p) \rightarrow (p.q))$ [*7, $A_2$ $q/p, r/q$; $R_1$]

*21 $(-(p.q) \rightarrow -(q.p))$ [*20, *15 $q/(q.p), p/(p.q)$; $R_1$].

Next we derive the traditional principle of *Denying the Disjunct*:

*22 $(-(p.q) \rightarrow -(p.q))$ [*7 $p/-(p.q)$]

*23 $((p.-(p.q)) \rightarrow -q)$ [*22, $A_3$ $p/-(p.q), q/p, r/q$; $R_1$].

And from this, by obvious steps, we can get various forms of *Denying the Alternant.* Neither the latter nor *23, it should be noted, are derivable as theorems in Ackermann's system of *strenge Implikation* [[1], [2]].

Next we get

*24 $((p.(p \rightarrow q)) \rightarrow --q)$ [$A_7$, $A_3$ $p/(p \rightarrow q), q/p, r/-q$; $R_1$],

from which, by *8, we get *Affirming the Antecedent*, and thereby any of the traditional variations of valid *Mixed Hypothetical Syllogisms.*

The *Principle of Pure Hypothetical Syllogisms* in $A_1$ can be supplemented by a second form of that principle, $\vdash ((p \to q) \to ((q \to r) \to (p \to r)))$, which is derived below using only $A_1$ and principles of transposition:

*25 $((-q \to -p) \to ((-r \to -q) \to (-r \to -p)))$ [$A_1$ $q/-q, r/-p, p/-r$]

*26 $((p \to q) \to (-q \to -p))$ [*15 $q/p, p/q$]

*27 $(((p \to q) \to (-q \to -p)) \to ((p \to q) \to ((-r \to -q) \to (-r \to -p))))$
[*25, $A_1$ $q/(-q \to -p), r/((-r \to -q) \to (-r \to -p)), p/(p \to q)$; $R_1$]

*28 $((p \to q) \to ((-r \to -q) \to (-r \to -p)))$ [*26, *27; $R_1$]

*29 $((-r \to -p) \to (p \to r))$ [*13 $p/r, q/p$]

*30 $(((-r \to -q) \to (-r \to -p)) \to ((-r \to -q) \to (p \to r)))$
[*29, $A_1$ $q/(-r \to -p), r/(p \to r), p/(-r \to -q)$; $R_1$]

*31 $(((p \to q) \to ((-r \to -q) \to (-r \to -p))) \to$
$\to ((p \to q) \to ((-r \to -q) \to (p \to r))))$
[*30, $A_1$ $q/((-r \to -q) \to (-r \to -p))$,
$r/((-r \to -p) \to (p \to r)), q/(p \to q)$; $R_1$]

*32 $((p \to q) \to ((-r \to -q) \to (p \to r)))$ [*28, *31; $R_1$]

*33 $((q \to r) \to (-r \to -q))$ [*15, $p/r$]

*34 $(-(-r \to -q) \to -(q \to r))$ [*33, *15 $q/(q \to r), p/(-r \to -q)$; $R_1$]

*35 $((-(p \to r) \to -(-r \to -q)) \to (-(p \to r) \to -(q \to r)))$
[*34, $A_1$ $q/-(-r \to -q), r/-(q \to r), p/-(p \to r)$; $R_1$]

*36 $((-(p \to r) \to -(q \to r)) \to ((q \to r) \to (p \to r)))$
[*13 $p/(p \to r), q/(q \to r)$]

*37 $(((-r \to -q) \to (p \to r)) \to (-(p \to r) \to -(-r \to -q)))$
[*15 $q/(-r \to -q), p/(p \to r)$]

*38 $((((-r \to -q) \to (p \to r)) \to (-(p \to r) \to -(-r \to -q))) \to$
$\to (((-r \to -q) \to (p \to r)) \to (-(p \to r) \to -(q \to r))))$
[*35, $A_1$ $q/(-(p \to r) \to -(-r \to -q)), r/(-(p \to r) \to -(q \to r))$,
$p/((-r \to -q) \to (p \to r))$; $R_1$]

*39 $(((-r \to -q) \to (p \to r)) \to (-(p \to r) \to -(q \to r)))$ [*37, *38; $R_1$]

*40 $((((-r \to -q) \to (p \to r)) \to (-(p \to r) \to -(q \to r))) \to$
$\to (((-r \to -q) \to (p \to r)) \to ((q \to r) \to (p \to r))))$
[*36, $A_1$ $q/(-(p \to r) \to -(q \to r)), r/((q \to r) \to (p \to r))$,
$p/((-r \to -q) \to (p \to r))$; $R_1$]

*41 $(((-r \to -q) \to (p \to r)) \to ((q \to r) \to (p \to r)))$ [*39, *40; $R_1$]

*42 $(((p \to q) \to (-r \to -q)) \to (p \to r))) \to$
$\to ((p \to q) \to ((q \to r) \to (p \to r))))$
[*41, $A_1$ $q/((-r \to -q) \to (p \to r)), r/((q \to r) \to (p \to r)), p/(p \to q)$; $R_1$]

*43 $((p \to q) \to ((q \to r) \to (p \to r)))$ [*32, *42; $R_1$].

Using principles of transposition, $A_1$ and *43 yield all varieties of *Pure Hypothetical Syllogisms* and traditional varieties of *Sorites*. Such principles can also be formulated with premisses in conjunction by using the *Principle of Importation*, which is derivable in seven steps (using *8, $A_1$, and $A_7$) from

*48 below:

*44 $((p.q) \rightarrow (q.p))$ [*20 $q/p, p/q$]
*45 $(((q.p) \rightarrow -r) \rightarrow ((p.q) \rightarrow -r))$ [*44, *43 $p/(p.q), q/(q.p), r/-r$; $R_1$]
*46 $((((q.p) \rightarrow -r) \rightarrow ((p.q) \rightarrow -r)) \rightarrow ((p \rightarrow -(q.r)) \rightarrow ((p.q) \rightarrow -r)))$
[$A_3$, *43 $p/(p \rightarrow -(q.r)), q/((q.p) \rightarrow -r), r/((p.q) \rightarrow -r)$; $R_1$]
*47 $((p \rightarrow -(q.r)) \rightarrow ((p.q) \rightarrow -r))$ [*45, *46; $R_1$]
*48 $((p \rightarrow -(q.-r)) \rightarrow ((p.q) \rightarrow --r))$ [*47 $r/-r$].

By theorem *52 below, together with $A_2$, $\vdash ((p \rightarrow q) \rightarrow ((p.r) \rightarrow (q.r)))$, and $\vdash ((p \rightarrow q) \rightarrow ((p.r) \rightarrow (r.q)))$ — which could have been added in seven steps — we can produce a complete set of *associative laws* for conjunction and alternation including '$(((p \vee (q \vee r)) \rightarrow (q \vee (p \vee r)))$' and '$(((p \supset (q \supset r)) \rightarrow (q \supset (p \supset r)))$'. We include only the proof of *52 however, since only *52 is needed in proofs below.

*49 $((q.r) \rightarrow (r.q))$ [*20 $p/r$]
*50 $(((r.p) \rightarrow (q.r)) \rightarrow ((r.p) \rightarrow (r.q)))$
[*49, $A_1$ $q/(q.r), r/(r.q), p/(r.p)$; $R_1$]
*51 $(((p \rightarrow q) \rightarrow ((r.p) \rightarrow (q.r))) \rightarrow ((p \rightarrow q) \rightarrow ((r.p) \rightarrow (r.q))))$
[*50, $A_1$ $q/((r.p) \rightarrow (q.r)), r/((r.p) \rightarrow (r.q)), p/(p \rightarrow q)$; $R_1$]
*52 $((p \rightarrow q) \rightarrow ((r.p) \rightarrow (r.q)))$ [$A_2$, *51; $R_1$].

From the provable principles above, Leibniz' *Praeclarum Principle*, i.e., '$(((p \rightarrow q).(r \rightarrow s)) \rightarrow ((p.r) \rightarrow (q.s)))$', can be derived and from this in turn all types of *Complex Dilemmas* follow.

Like most modern formalizations of propositional logic, $P_{A1}$ can be extended to cover many valid arguments of greater complexity than those handled in traditional logic. But the proofs above suffice to show that $P_{A1}$ is broad enough in scope to be generally compatible with traditional logic.

**4. Completeness of $P_{A1}$ with respect to *Principia mathematica*.** Every theorem of the propositional calculus in PM is also a theorem in $P_{A1}$. First we establish:

$$(p \supset (p.p))$$
$$((p.q) \supset p)$$
$$((p \supset q) \supset (-(q.r) \supset -(r.p))),$$

which constitute an axiom set of Rosser's [13]; then we show that $P_{A1}$ contains identical or equivalent rules and elements to those of Rosser's system.

The first two of these formulas follow directly from $A_9$, $A_8$, and $D_2$.

*53 $(p \supset (p.p))$ [$A_9$, $D_2$; $R_4$]
*54 $((p.q) \supset p)$ [$A_8$, $D_2$; $R_4$].

The third formula, however, requires a longer proof:

*55 $(-(q.r) \rightarrow -(r.q))$ [*21 $p/q, q/r$]
*56 $((-(q.r) \rightarrow -(r.q)) \rightarrow ((-(q.r).r) \rightarrow -q))$ [*47 $p/-(q.r), q/r, r/q$]
*57 $((-(q.r).r) \rightarrow -q)$ [*55, *56; $R_1$]
*58 $((p.-(p.-q)) \rightarrow --q)$ [*23 $q/-q$]
*59 $(-q \rightarrow -(p.-(p.-q)))$ [*58, $A_5\, p/(p.-(p.-q)), q/-q$; $R_1$]
*60 $((-q \rightarrow -(p.-(p.-q))) \rightarrow ((-(q.r).r) \rightarrow -(p.-(p.-q))))$
[*57, *43 $p/(-(q.r).r), q/-q, r/-(p.-(p.-q))$; $R_1$]
*61 $(((-(q.r).r) \rightarrow -(p.-(p.-q)))$ [*59, *60; $R_1$]
*62 $((p.(-(q.r).r)) \rightarrow --(p.-q))$
[*61, $A_3\, p/(-(q.r).r), q/p, r/-(p.-q)$; $R_1$]
*63 $(-(q.r).(p.r)) \rightarrow (p.(-(q.r).r)))$ [$A_4\, p/-(q.-r), q/p$]
*64 $(((p.(-(q.r).r)) \rightarrow --(p.-q)) \rightarrow ((-(q.r).(p.r)) \rightarrow --(p.-q)))$
[*63, *43 $p/(-(q.r).(p.r)), q/(p.(-(q.r).r)),$
$r/--(p.-q)$; $R_1$]
*65 $((-(q.r).(p.r)) \rightarrow --(p.-q))$ [*62, *64; $R_1$]
*66 $(-(p.r) \rightarrow -(r.p))$ [*21 $q/r$]
*67 $(--(r.p) \rightarrow (p.r))$ [*66, *19 $q/(p.r), p/-(r.p)$; $R_1$]
*68 $((-(q.r).--(r.p)) \rightarrow (-(q.r).(p.r)))$
[*67, *52 $p/--(r.p), q/(p.r), r/-(q.r)$; $R_1$]
*69 $(((-(q.r).(p.r)) \rightarrow --(p.-q)) \rightarrow ((-(q.r).--(r.p)) \rightarrow --(p.-q)))$
[*68, *43 $p/(-(q.r).--(r.p)), q/(-(q.r).(p.r)),$
$r/--(p.-q)$; $R_1$]
*70 $((-(q.r).--(r.p)) \rightarrow --(p.-q))$ [*65, *69; $R_1$]
*71 $(-(p.-q) \rightarrow -(-(q.r).--(r.p)))$
[*70, $A_5\, p/(-(q.r).--(r.p)), q/-(p.-q)$; $R_1$]
*72 $((p \supset q) \rightarrow (-(q.r) \supset -(r.p)))$ [*71, $D_2$; $R_4$]
*73 $-((p \supset q).-(-(q.r) \supset -(r.p)))$
*[72, $A_7\, p/(p \supset q), q/(-(q.r) \supset -(p.r))$; $R_1$]
*74 $((p \supset q) \supset (-(q.r) \supset -(r.p)))$ [*73, $D_2$; $R_4$].

Rosser's rule of inference "If $\vdash S$ and $\vdash (S \supset S')$, then $\vdash S'$" is established as a derived rule of inference in $P_{A1}$ by virtue of $R_1$, $R_2$, and *76 below:

*75 $((p.-(q.-p)) \rightarrow q)$ [*58, *8 $q/(p.-(p.-q)), p/q$; $R_1$]
*76 $((p.(p \supset q)) \rightarrow q)$ [*75, $D_2$; $R_4$].

Since all other elements of the Rosser system are contained in $P_{A1}$, and the Rosser system has been shown equivalent to the propositional calculus of PM [[6], Ch. 7], it follows that $P_{A1}$ is complete with respect to PM.

Since the interpretation of '$\supset$' is restricted to expressions involving only conjunction (or alternation) and denial in $P_{A1}$, the inclusion of the PM calculus does not entail the inclusion of "paradoxes of material, or strict, implication." Theorems in this portion of $P_{A1}$ will contain no logically true implications and hence indicate no patterns of valid argument; they consist solely of logically true disjunctions, alternations, or (rarely) conjunctions, all of which are categorical, not hypothetical.

**5. Additional subjunctive theorems of $P_{A1}$.** Axioms 1-9, and all theorems derived from these axioms alone, are compatible with PM in the following sense: if '$\rightarrow$' were replaced by '$\supset$' at all occurrences, the resulting formulas would all be theorems in the PM calculus of propositions. In the same sense, Lewis's system of "strict implication" [[10] Ch. VI and Appendix II] and Ackermann's system of "strenge Implikation" [1] — ignoring theorems containing modal operators or quantifiers — as well as Heyting's intuitionist logic and most other formalized propositional logics are compatible with PM when their respective conditional signs are replaced throughout by '$\supset$'. Such systems, in short, contain restricted sub-classes of the conditional theorems sanctioned in PM, without adding any new theorems. The system $P_{A1}$. however, contains in Axiom 10 the source of an infinite number of theorems which are not provable in PM, and which, in some cases, are logical contradictories of theorems which would appear in each of the systems mentioned above.

Consider first the formula '$-(p \rightarrow -p)$' which is proved as a theorem of $P_{A1}$:

*77 $\quad -(p \rightarrow -p)$ $\qquad\qquad$ [*7, $A_{10}\ q/p$; $R_1$].

Interpreting '$\rightarrow$' as the subjunctive "if ... then ...", as specified in condition 6, this theorem may be interpreted as asserting "It is false that if $p$ were true, then $p$ would be false". Intuitively, this would seem an appropriate candidate for a logically necessary subjunctive conditional. But the '$\supset$'– for –'$\rightarrow$' analogue, i.e., '$-(p \supset -p)$', is not a theorem in PM, since '$-(p \supset -p)$' is equivalent to '$--(p.--p)$', hence to '$p$'. Indeed, "$-$(If $p$ then $-p$)" cannot be a theorem in any of the three other systems (those of Lewis, Ackermann, and Heyting). For in each of these systems, as well as in PM, one can prove $\vdash$ (If $(p.-p)$ then $-(p.-p)$), which is a substitution instance of "If $p$ then $-p$", and is thus inconsistent with *77. Since $P_{A1}$ is consistent "$((p.-p) \rightarrow -(p.-p))$" cannot be a theorem in it, whence it follows that $P_{A1}$ is independent of each of the four systems mentioned, and also that the treatment of "if ... then ..." in $P_{A1}$ can neither be reduced to, nor reconciled with, conditionals in these other systems.

Axiom 10 is considerably stronger than *77, however. It says, in effect, that if a proposition $p$ implies another, $q$, then $p$ can not imply the contradictory of $q$. Three variants of this principle are established below as *78, *79, and *80:

*78 $((p \rightarrow -q) \rightarrow -(p \rightarrow q))$ [$A_{10}$, $A_5$ $p/(p \rightarrow q)$, $q/(p \rightarrow -q)$; $R_1$]
*79 $((-p \rightarrow q) \rightarrow -(-p \rightarrow -q))$ [$A_{10}$, $p/-p$]
*80 $((-p \rightarrow -q) \rightarrow -(-p \rightarrow q))$ [*78 $p/-p$].

From *80, incidently, we get *82, which denies the converse of the conditional in *77:

*81 $(-p \rightarrow -p)$ [*7 $p/-p$]
*82 $-(-p \rightarrow p)$ [*81, *80 $q/p$; $R_1$].

Other variations of the new principle are introduced by laws of transposition. One variation, like *86 below, says in effect that if an antecedent implies a certain consequent, that consequent will not imply the contradictory of that antecedent:

*83 $((q \rightarrow -p) \rightarrow (p \rightarrow -q))$ [$A_5$ $p/q$, $q/p$]
*84 $(-(p \rightarrow -q) \rightarrow -(q \rightarrow -p))$ [*83, *15 $q/(q \rightarrow -p)$, $p/(p \rightarrow -q)$; $R_1$]
*85 $((-(p \rightarrow -q) \rightarrow -(q \rightarrow -p)) \rightarrow ((p \rightarrow q) \rightarrow -(q \rightarrow -p)))$
[$A_{10}$, *43 $p/(p \rightarrow q)$, $q/-(p \rightarrow -q)$, $r/-(q \rightarrow -p)$; $R_1$]
*86 $((p \rightarrow q) \rightarrow -(q \rightarrow -p))$ [*84, *85; $R_1$].

A second variation introduced by transposition, says that if a certain antecedent implies a certain consequent, the denial of the consequent will not imply the antecedent; e.g., *90 below:

*87 $((p \rightarrow q) \rightarrow (-q \rightarrow -p))$ [*15 $q/p$, $p/q$]
*88 $(((-q \rightarrow -p) \rightarrow -(-q \rightarrow p)) \rightarrow ((p \rightarrow q) \rightarrow -(-q \rightarrow p)))$
[*87, *43 $p/(p \rightarrow q)$, $q/(-q \rightarrow -p)$, $r/-(-q \rightarrow p)$; $R_1$]
*89 $((-q \rightarrow -p) \rightarrow -(-q \rightarrow p))$ [*80 $p/q$, $q/p$]
*90 $((p \rightarrow q) \rightarrow -(-q \rightarrow p))$ [*89, *88; $R_1$].

Still another set of principles, like *92 below, is obtained from principles like $A_{10}$, *86, and *90 by inserting the latter as antecedents in *78:

*91 $((p \rightarrow q) \rightarrow -(p \rightarrow -q)) \rightarrow -((p \rightarrow q) \rightarrow (p \rightarrow -q))$
[*78, $p/(p \rightarrow q)$, $q/(p \rightarrow -q)$]
*92 $-((p \rightarrow q) \rightarrow (p \rightarrow -q))$ [$A_{10}$, *91; $R_1$].

These principles deny that a conditional can imply another conditional which has the properties proscribed in the previous principles. Finally, using $A_7$ we can derive various sets of disjunctive theorems which deny that certain pairs of conditionals can be true simultaneously. For example, *98:

*93 $((r.p) \rightarrow (r.--p))$ [*5, *52 $q/--p$; $R_1$]
*94 $(-(r.--p) \rightarrow -(r.p))$ [*93, *15 $q/(r.p)$, $p/(r.--p)$; $R_1$]
*95 $(((r \rightarrow -p) \rightarrow -(r.--p)) \rightarrow ((r \rightarrow -p) \rightarrow -(r.p)))$
[*94, $A_1$ $q/-(r.--p)$, $r/-(r.p)$, $p/(r \rightarrow -p)$; $R_1$]

| | | |
|---|---|---|
| *96 | $((r \rightarrow -p) \rightarrow -(r \,.\, --p))$ | $[A_7, p/r, q/-p]$ |
| *97 | $((r \rightarrow -p) \rightarrow -(r \,.\, p))$ | $[{*96}, {*95}; R_1]$ |
| *98 | $-((p \rightarrow q) \,.\, (p \rightarrow -q))$ | $[A_{10}, {*97}\, p/(p \rightarrow q), r/(p \rightarrow -q); R_1]$. |

That no analogues of these new principles appear in PM can be shown by replacing the '$\rightarrow$' by '$\supset$' throughout, then finding the truth-table of each when the antecedent in the first conditional is false and the consequent in the first conditional is true. In fact, in PM no conditionals in which either the consequent or the antecedent is consistent but non-tautological can be logically incompatible with each other and, indeed, no such conditionals can be logically false at all.

Our chief purpose in this paper is to establish the consistency of principles based on $A_{10}$ with traditional principles based on $A_1$–$A_9$; but the justification of adding $A_{10}$ must eventually be decided on other grounds. The appropriateness of the stronger principles is most easily approached by considering whether, in ordinary language, the conjunction of 'If $p$ then $q$' and 'If $p$ then not $q$' would be considered logically false (as asserted in *98). Provided the "if ... then ..." is subjunctive, an affirmative answer seems at least plausible. Surely one would ordinarily say of such pairs of conditionals as

1) If the match had been scratched, it would have lighted
1′) If the match had been scratched, it would not have lighted,
2) If we had followed a different policy towards Germany in the 1920s, the second World War would not have occurred
2′) If we had followed a different policy towards Germany in the 1920s, the second World War would still have occurred,

that they are conflicting, logically incompatible statements.

While we shall not attempt to establish the point conclusively here, this position is reinforced by various writers. Nelson [11], in trying to set up a logic of intension, formulated a postulate set from which he derived both $\vdash -(p \mathrm{E} -p)$ and $\vdash ((p \mathrm{E} q) \mathrm{E} -(p \mathrm{E} -q))$, where 'E' represents "entails," which was his alternative to the versions of "if ... then ..." in PM and in Lewis's "strict implication" [see [11], p. 449]. More recently, philosophical analysts appear to believe that ordinary language supports this analysis of "if ... then ....". For example, Strawson [14] writes,

> "The formulae '$p \supset q$' and '$p \supset -q$' are consistent with one another, and the joint assertion of corresponding statements of these forms is equivalent to the assertion of the corresponding statement of the form '$-p$'. But 'If it rains the match will be cancelled' is inconsistent with 'If it rains, the match will not be cancelled', and their joint assertion in the same context is self-contradictory." [1]

[1] P.F. Strawson, ***Introduction to logical theory***, p. 85.

And another writer [7] says,

> "It seems clear, though it is perhaps impossible to prove, that two subjunctive conditionals with the same antecedent and contradictory consequents cannot both be true." [2]

If one is satisfied with the disjunction in *98, one should have little objection to the subjunctive hypothetical "If $(p \rightarrow q)$ then $-(p \rightarrow -q)$" of $A_{10}$.

The objection has been raised that such theorems as $A_{10}$, *77, and *98 would eliminate *Reductio ad Absurdum* proofs. The objection raises questions, but it is clear that no proof procedure of PM or its logistic foundations of mathematics would thereby be threatened. For, since $P_{A1}$ is complete with respect to PM, such laws as Church's [[5], p. 142],

$((p \supset -p) \supset -p)$ (Special Law of *Reductio ad Absurdum*)
$((p \supset q) \supset ((p \supset -q) \supset -p)$ (Law of *Reductio ad Absurdum*),

though not interpretable as involving conditionals, are theorems in $P_{A1}$; and from them, with $R_1$, $R_2$, and *76 we can get the derived rules of inference:

If $\vdash (S \supset -S)$ then $\vdash -S$
If $\vdash (S \supset S')$ and $\vdash (S \supset -S')$ then $\vdash -S$.

Further, there is at least one standard schema for *Reductio ad Absurdum* proofs, namely,

If $p$ then ($q$ and $-q$)
Hence, not $p$,

which can be formalized as the derived rule of inference in $P_{A1}$,

If $\vdash (S \rightarrow (S' . -S'))$, then $\vdash -S$,

by virtue of $R_1$, $R_2$, and '$-(p . -p)$' and '$(((p \rightarrow q) . -q) \rightarrow -p)$' which are theorems in $P_{A1}$. On the other hand, $P_{A1}$ does not admit the '$\rightarrow$'– for –'$\supset$' analogues of the first two theorems above (let $p = 0$, $q = 0$), or of the related derived rules; but this fact is counted a defect in $P_{A1}$ only if it is held that non-material conditionals are essential in the proper formulation of *Reductio ad Absurdum* along the lines of these theorems.

If Axiom 10 and its consequences be admitted, then the traditional hypothetical theorems established by $A_1$ to $A_9$ are supplemented by an indefinite class of co-related, negative theorems. For example:

From *Commutation* or *Permutation* we get,

*99 $-((p . q) \rightarrow -(q . p))$ [*20, *86 $p/(q . p)$, $q/(p . q)$; $R_1$]
*100 $-(-(p . q) \rightarrow (q . p))$ [*21, *80 $p/(p . q)$, $q/(q . p)$; $R_1$].

---

[2] P. B. Downing, *Subjunctive conditionals, time order and causation*, **Proceedings of the Aristotelian Society**, n.s. vol. LIX (1959), p. 126.

From *Denying the Disjunct* we get,

$$*101 \quad -((p.-(p.q)) \rightarrow q) \qquad [*23, *78\ p/(p.-(p.q)), q/q; R_1].$$

And from the principle of the *Pure Hypothetical Syllogism* we can get, not only

$$*102 \quad -((p \rightarrow q) \rightarrow -((q \rightarrow r) \rightarrow (p \rightarrow r))$$
$$[*43, A_{10}\ p/(p \rightarrow q), q/((q \rightarrow r) \rightarrow (p \rightarrow r)); R_1],$$

but also such theorems as

$$((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow -(p \rightarrow -r)))$$
$$-((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow -r)))$$
$$((p \rightarrow q) \rightarrow -((q \rightarrow r) \rightarrow (p \rightarrow -r))),$$

and so on.

PM has frequently been charged with including too much, e.g., the "paradoxes of material implication"; one wonders why it has not been charged with inability to include analogues of the theorems above.

**6. Elimination of "paradoxes" of material and strict implication.** It is agreed that these so-called "paradoxes" are not really logical paradoxes at all. In Nelson's words:

> "The so-called paradoxical propositions both of material and of strict implication are in terms of the respective systems not paradoxes at all. It is only when we are told that the symbols '$\supset$' and '$\prec$' represent what is commonly understood by the word 'implication', that these propositions appear paradoxical." [3]

For this reason, the inclusion of the PM calculus in $P_{A1}$ does not entail that $P_{A1}$ contains anything paradoxical. What must be considered, however, is whether '$\rightarrow$' does not give rise to analogous paradoxes arising from the fact that '$\rightarrow$' *is* interpreted as "if-then."

It is easy to show the non-derivability in $P_{A1}$ of the '$\rightarrow$'– for –'$\supset$' analogues of any of the PM theorems which Lewis [[9], pp. 325–6 and [10], pp. 141–5] identified as "paradoxes of material implication." For example, '$(-p \supset (p \supset q))$' has as its analogue '$(-p \rightarrow (p \rightarrow q))$', and in the truth-tables given above the latter formula comes out false when $p = 0$, $q = 1$.

The "paradoxes" of Lewis's system of strict implication were stated by Lewis to "become explicit" in his four theorems [[10], pp. 174–5]:

19.74 $-\Diamond p . \prec . p \prec q$ ("A proposition which is self-contradictory or impossible, implies any proposition")

19.75 $-\Diamond - p . \prec . q \prec p$ ("A proposition which is necessarily true is implied by any proposition")

[3] Everett J. Nelson, *Intensional relations*, **Mind,** n.s. vol. 39 (1930), p. 448.

19.76 $-(p \prec q) \prec \Diamond p$ ("If there is any proposition $q$ which $p$ does not imply, then $p$ is self-consistent or possible")

19.77 $-(q \prec p) \prec \Diamond -p$ ("If there is any proposition $q$ which does not imply $p$, then $p$ is possibly false").

Since $P_{A1}$ does not contain modal operators, we shall handle these four "paradoxes" in terms of four implied meta-logical assertions. Taking theorems of logic as "necessary," and denials of theorems as "impossible," we construe 19.74 to 19.77 as implying, respectively, the following meta-logical assertions with respect to $P_{A1}$:

1) Every wff of the form $(S \rightarrow S')$, where S is the denial of a theorem and S′ is any wff whatever, is a theorem.
2) Every wff of the form $(S \rightarrow S')$, where S′ is a theorem and S is any wff whatever, is a theorem.
3) If some wff $(S \rightarrow S')$ is not a theorem, then $-S$ is not a theorem.
4) If some wff $(S \rightarrow S')$ is not a theorem, then S′ is not a theorem.

These assertions are proved false of $P_{A1}$ by the following counter-examples: 1) Let S be '$-(p \rightarrow p)$' and S′ be '$q$'; 2) Let S′ be '$(p \rightarrow p)$' and S be '$q$'; 3) Let S be '$(p \rightarrow -p)$' and S′ be '$q$'; 4) Let S be '$q$' and S′ be '$(p \rightarrow p)$'.

More generally, consideration of the matrix for '$\rightarrow$' in section 2 shows that *no* theorem is implied by every proposition and *no* denial of any theorem implies every proposition. Indeed, consideration of $A_{10}$ and *98 makes it plain that *no well-formed formula*, theorem or not, can imply every proposition in $P_{A1}$.

While the discussion above proves that $P_{A1}$ does not include *all* paradoxes of material and strict implication, and does not include any of those particular "paradoxes" which Lewis (and others) consider central and most significant, it has not been established that $P_{A1}$ does not include *any* such "paradoxes." The proof of the latter, however, could only be established if the notion of such "paradoxes" were formulated in purely formal terms. As the "paradoxes" are semantic rather than formal, this seems an unlikely prospect. (For two attempts at partial formalization see Anderson [2] and Belnap [3]).

Other evidence that $P_{A1}$ avoids the "paradoxes of implication" lies in the fact that certain formulas, described by some as the "sources" of these paradoxes, are non-derivable in $P_{A1}$. These are discussed in the following section.

**7. Residual problems.** It can be argued that $P_{A1}$ is superior to standard calculi as a logic of propositions in that 1) it retains the full deductive power of PM, requiring only a generally accepted restriction of interpretation, 2) it includes the major principles of traditional logic which involve "if ... then ...," 3) it clears up the awkward "paradoxes", and

4) it includes a set of additional principles which appear compatible with, and perhaps necessary for, the interpretation of the conditional sign as subjunctive. Though $P_{A1}$ is not as elegant as some standard calculi, it can be urged that this is outweighed by its extended suitability (i.e., for informal uses of "if-then," as well as for the tasks served by PM).

But all this is not enough. Residual problems arise concerning whether certain formulas *should* be theorems but are not, or whether others *should not* be theorems though they are. The most obvious of these we discuss below. In certain cases we defend $P_{A1}$, but there are certain results which, in this writer's opinion, make $P_{A1}$ less than completely satisfactory.

Each of the wffs below are analogues of theorems in PM and related systems, and they have been advanced not solely as syntactical conveniences but also as representing analytic truths in ordinary language.

| | | |
|---|---|---|
| (1) Simplification: | $((p.q) \rightarrow p)$ | $[p = 0, q = 0]$ |
| (2) Contrapositive of (1): | $(-p \rightarrow -(p.q))$ | $[p = 0, q = 0]$ |
| (3) Exportation: | $(((p.q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r)))$ | $[p = 0, q = 0, r = 1]$ |
| (4) Assertion: | $(p \rightarrow ((p \rightarrow q) \rightarrow q))$ | $[p = 0, q = 0]$ |
| (5) Commutation: | $((p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)))$ | $[p = 1, q = 0, r = 0]$ |
| (6) Adjunction: | $(p \rightarrow (q \rightarrow (p.q)))$ | $[p = 0, q = 0]$ |
| (7) Tautology: | $(p \rightarrow (p.p))$ | $[p = 0]$. |

None of these formulas is a theorem in $P_{A1}$, as value assignments on the right show.

The omission of (1), Simplification, is necessary; for with '$-p$' in place of '$q$' Simplification would conflict with *98 by producing both '$((p.-p) \rightarrow p)$' and '$((p.-p) \rightarrow -p)$', and with Transposition (*15) and $A_3$ it would yield the "paradox," '$((p.-p) \rightarrow q)$'. Apart from such consequences, it could be argued that Simplification is not universal because ' If $(p.-p)$ were true then $p$ would be true," is not an inescapable logical truth; should someone assert "$p$ and not $p$" we should scarcely assert that the truth of $p$ was implicitly intended. Along different lines, Nelson argued that $p$ is not implied by $p$ *and* $q$ (which should be viewed as a whole) but only by one component, $p$, with respect to which '*and* $q$' is superfluous [[11], pp. 447–48]; this suggests a notion of "if-then," apparently compatible with $P_{A1}$, in which one component is properly said to be *a condition* of another only if all parts of the antecedent are directly or indirectly contributory to the consequent. Whether such arguments will stand up under continued philosophical analysis may be questioned, but they clearly suggest the possibility of reasons, other than syntactical, for excluding Simplification. This suggestion may be rendered more palatable by the reminder that $P_{A1}$ still retains, by virtue of *54 and *76, the derived rule of inference, "If $\vdash$ (S.S′), then $\vdash$ S′," and thus its deductive power is not lessened with respect to PM.

It might be argued that Simplification should be excluded, but its contrapositive, (2) $(-p \rightarrow -(p.q))$, admitted; since it seems inescapable that if $p$ were false, then the conjunction of $p$ with anything else would have to be false. If this were granted then we should get, as well,

(8) $(-p \rightarrow (p \supset q))$, (9) $(p \rightarrow (-p \supset q))$, (10) $(p \rightarrow (q \vee p))$, etc.,

which are not in themselves "paradoxes" since '$\supset$' is not interpreted as "if-then." But to get these and exclude Simplification we should have to eliminate $A_3$ and replace $A_5$ by '$((-p \rightarrow q) \rightarrow (-q \rightarrow p))$'. This would lead to the exclusion of an even larger class of generally accepted theorems.

The *Principle of Exportation* would make '$\rightarrow$' equivalent to material implication, and this would lead to inconsistency.

| | | |
|---|---|---|
| (i) | $((p \supset q) \rightarrow -(p.-q))$ | [*7, $p/-(p.-q)$, $D_2$; $R_4$] |
| (ii) | $(((p \supset q).p) \rightarrow --q)$ | [(i), *48 $p/(p \supset q)$, $q/p$, $r/q$; $R_1$] |
| (iii) | $(((p \supset q).p) \rightarrow q)$ | [(ii), *8 $q/((p \supset q).p)$, $p/q$; $R_1$] |
| (iv) | $((((p \supset q).p) \rightarrow q) \rightarrow ((p \supset q) \rightarrow (p \rightarrow q)))$ | [*Exportation*, $p/(p \supset q)$, $q/p$, $r/q$] |
| (v) | $((p \supset q) \rightarrow (p \rightarrow q))$ | [(ii), (iii); $R_1$] |
| (vi) | $((p \rightarrow q) \rightarrow (p \supset q))$ | [$A_7$, $D_2$; $R_4$]. |

But the omission of Exportation apparently presents little difficulty as it is non-derivable in the systems of Ackermann, Burks, Nelson, and Lewis, and little or no objection has been raised.

Somewhat more difficult to defend is the omission of (4) Assertion and (5) Commutation. We consider these together since, given either one, the other may be derived in a few steps, using *43 with Assertion to get Commutation, or *7 [with $p/(p \rightarrow q)$] and Commutation to get Assertion. Though independent of $P_{A1}$, it is not clear that these two are incompatible with it; their non-inclusion is based partly on the fact that the writer has found no proof of their compatibility with $A_1$ to $A_{10}$. Assertion appears attractive because it parallels certain formulations of the rule *Modus Ponens*. But in $P_{A1}$ the theorem, '$((p \rightarrow q) \rightarrow (p \rightarrow q))$' [by *7 $p/(p \rightarrow q)$], is quite as serviceable. The statement, "If $(p \rightarrow q)$ were true, then if $p$ were true, $q$ would be true," is as adequate for that purpose as "If $p$ were true, then if $(p \rightarrow q)$ were true, then $q$ would be true." As for Commutation, we shall see below that while its admission with $A_1$ to $A_{10}$ might be consistent, its admission with (6) Adjunction — in many ways a more desirable formula — would reduce '$\rightarrow$' to material implication.

There are some serious disadvantages to the omission of (6) Adjunction, Its exclusion is responsible for the presence of the rule of transformation, $R_2$, which would be unnecessary if Adjunction were a theorem. Also, if Adjunction were included, hypothetical formulations of "denying the alternant" and related principles would result by way of Transposition and Syllogism:

(11) $(p \rightarrow (-(p.q) \rightarrow -q))$, (12) $(-p \rightarrow ((p \vee q) \rightarrow q)$,
(13) $(p \rightarrow ((p \supset q) \rightarrow q))$.

Finally, like (7), Adjunction has a strong intuitive appeal as an analytic truth. Although the independence of these principles is clear, the writer has found no proof that they are incompatible with $P_{A1}$. What is clear, however, is that if Adjunction were admitted, Commutation and Assertion could not be. For from (13) above and Commutation we would get $((p \supset q) \rightarrow (p \rightarrow q))$.

The omissions just discussed are not as serious as they might appear for several reasons. In the first place, the '$\supset$'–for–'$\rightarrow$' analogues of (1)–(13) appear as theorems in $P_{A1}$ with only a restriction on interpretation. Secondly, derived rules of inference in which such theorems act as premisses will be coextensive with such rules for PM. And finally, many of the excluded formulas have close approximations among the theorems of $P_{A1}$; e.g., $P_{A1}$ lacks Exportation but includes '$(((p.q) \rightarrow r) \rightarrow (p \rightarrow (q \supset r)))$' as a theorem. But while these considerations mitigate the difficulties they do not remove them. And in addition there are formulas which appear as theorems which might be subject to objections similar to those used to justify omissions above. For example, by substitution of '$p$' for '$q$' in *23 we get '$((p.-(p.p)) \rightarrow -p)$' as a theorem. While this is far removed from the "paradoxes," it might be criticized on the grounds that there is no particular reason why the inconsistent antecedent should "imply" the particular consequence it does.

The difficulties just discussed might conceivably be argued away, but they seem sufficient to this writer to show that $P_{A1}$ is not a completely satisfactory formalization of the logic of propositions, whether or not it is more so than present alternatives.

Apart from merits or defects of $P_{A1}$, however, its existence demonstrates the feasibility of a new approach to the logic of propositions involving the *principle of subjunctive contrariety*. We thus have good reason to investigate the effect this principle, and a concept of conditionality compatible with it, might exert if introduced into standard quantification theory, into set theory, into modal logic and into epistemology and the philosophy of science.

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